NAMUNAVIY MASALALAR

1-misol (kasrli Koshi)

Ijobiy a, b, c sonlar uchun isbotlang:

$$a/(b + c) + b/(c + a) + c/(a + b) \geq 3/2$$

Yechish

Koshi tengsizligining Engel ko‘rinishi:

$$x_1{}^2/y_1 + x_2{}^2/y_2 + x_3{}^2/y_3 \geq (x_1 + x_2 + x_3)^2/(y_1 + y_2 + y_3)\ (y_i > 0)$$

Bu yerda $x_1 = a, y_1 = a(b + c), x_2 = b, y_2 = b(c + a), x_3 = c, y_3 = c(a + b)$ deb olamiz.

Shunda:

$$a/(b + c) = a^2/(a(b + c))$$

Demak,

$$\begin{aligned} &a/(b + c) + b/(c + a) + c/(a + b) \\ &= a^2/(a(b + c)) + b^2/(b(c + a)) + c^2/(c(a + b)) \\ &\geq (a + b + c)^2/(a(b + c) + b(c + a) + c(a + b)) \end{aligned}$$

Pastdagi yig‘indini soddalashtiramiz:

$$\begin{aligned} &a(b + c) + b(c + a) + c(a + b) \\ &= ab + ac + bc + ab + ac + bc \\ &= 2(ab + ac + bc) \end{aligned}$$

Demak,

$$\begin{aligned} a/(b + c) + b/(c + a) + c/(a + b) \\ \geq (a + b + c)^2/(2(ab + ac + bc)) \end{aligned}$$

Endi ma’lum tengsizlik:

$$(a + b + c)^2 \geq 3(ab + ac + bc)$$

Shuni qo‘llasak:

$$\begin{aligned} (a + b + c)^2/(2(ab + ac + bc)) \\ \geq 3(ab + ac + bc)/(2(ab + ac + bc)) = 3/2 \end{aligned}$$

Tengsizlik isbotlandi.

2-misol (maksimum, vektor uslubida)

Agar $a^2 + b^2 + c^2 = 1$ bo‘lsa, $a + b + c$ ning maksimal qiymatini toping.

Yechish

Koshi tengsizligi:

$$(a + b + c)^2 \leq (a^2 + b^2 + c^2) \cdot (1^2 + 1^2 + 1^2)$$

$$(a + b + c)^2 \leq 1 \cdot 3 = 3$$

Demak:

$$a + b + c \leq \sqrt{3}$$

Javob

Maksimal qiymat $\sqrt{3}$.

3-misol (klassik baholash)

Haqiqiy a, b sonlar uchun isbotlang:

$$(a^2 + b^2)(1 + 1) \geq (a + b)^2$$

Yechish

Bu Koshi tengsizligining aynan o‘zi:

$$(a \cdot 1 + b \cdot 1)^2 \leq (a^2 + b^2) \cdot (1^2 + 1^2)$$

$$(a + b)^2 \leq 2(a^2 + b^2)$$

Yoki qayta yozsak:

$$(a^2 + b^2) \cdot 2 \geq (a + b)^2$$

Xulosa

Isbotlandi.

(Bu masala murakkab emasdek ko‘rinadi, lekin keyingi masalalarda shu “baholash skeleti” juda ko‘p ishlaydi.)

4-misol (Engel ko‘rinishidan bevosita)

Ijobiy a, b, c uchun isbotlang:

$$a^2/(b + c) + b^2/(c + a) + c^2/(a + b) \geq (a + b + c)/2$$

Yechish

Engel ko‘rinishi:

$$a^2/(b + c) + b^2/(c + a) + c^2/(a + b)$$
$$\geq (a + b + c)^2/((b + c) + (c + a) + (a + b))$$

Pastki qism:

$$(b + c) + (c + a) + (a + b) = 2(a + b + c)$$

Demak:

$$\geq (a + b + c)^2/(2(a + b + c)) = (a + b + c)/2$$

Xulosa

Tengsizlik isbotlandi.

5-misol (AM–GM bilan bog‘langan tadbiq)

Ijobiy a, b sonlar uchun isbotlang:

$$(a^2/b + b^2/a) \geq a + b$$

Yechish

Koshi tengsizligidan foydalanamiz:

$$(a^2/b + b^2/a) \cdot (a + b) \geq (a + b)^2$$

Ikkala tomonni $(a + b)$ ga bo‘lamiz:

$$a^2/b + b^2/a \geq a + b$$

Xulosa

Isbotlandi.

6-misol

Agar $a^2 + b^2 + c^2 = 1$ bo‘lsa,
$a + b + c \leq ?$ ni toping.

Yechish

Koshi tengsizligini (a, b, c) va $(1, 1, 1)$ uchun qo‘llaymiz:

$$(a + b + c)^2 \leq (a^2 + b^2 + c^2) \cdot (1 + 1 + 1)$$

$$(a + b + c)^2 \leq 1 \cdot 3 = 3$$

Demak:

$$a + b + c \leq \sqrt{3}$$

7-misol

Isbotlang $(a, b, c > 0)$:

$$a^2/(b + c) + b^2/(c + a) + c^2/(a + b) \geq (a + b + c)/2$$

Yechish

Koshi tengsizligining Engel ko‘rinishi:

$$a^2/(b + c) + b^2/(c + a) + c^2/(a + b)$$
$$\geq (a + b + c)^2/((b + c) + (c + a) + (a + b))$$

Pastki qism:

$$(b + c) + (c + a) + (a + b) = 2(a + b + c)$$

Shundan:

$$\geq (a + b + c)^2/(2(a + b + c)) = (a + b + c)/2$$

Xulosa

Isbotlandi.

8-misol

Agar $a^2 + b^2 = 5$ bo‘lsa,
$2a - b$ ifodaning maksimal qiymatini toping.

Yechish

Koshi tengsizligi (a, b) va $(2, -1)$ uchun:

$$(2a - b)^2 \leq (a^2 + b^2) \cdot (2^2 + (-1)^2)$$

$$(2a - b)^2 \leq 5 \cdot 5 = 25$$

Demak:

$$2a - b \leq 5$$

Javob

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9-misol

Isbotlang:

$$(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$$

Yechish

Koshi tengsizligini (a, b, c) va $(1, 1, 1)$ uchun qo‘llaymiz:

$$(a + b + c)^2 \leq (a^2 + b^2 + c^2) \cdot 3$$

Xulosa

Tengsizlik isbotlandi.

10-misol

Ijobiy a, b, c uchun isbotlang:

$$a/(b + c) + b/(c + a) + c/(a + b) \geq 3/2$$

Yechish

$a/(b + c) = a^2/(a(b + c))$ deb yozamiz va Engel ko‘rinishini qo‘llaymiz:

$$\geq (a + b + c)^2/(a(b + c) + b(c + a) + c(a + b))$$

Pastki qism:

$$= 2(ab + bc + ca)$$

Ma’lumki:

$$(a + b + c)^2 \geq 3(ab + bc + ca)$$

Shundan:

$$\geq 3(ab + bc + ca)/(2(ab + bc + ca)) = 3/2$$

Xulosa

Isbotlandi.

11-misol

Agar $a^2 + b^2 + c^2 = 6$ bo‘lsa,
$a + 2b + 2c$ ning maksimal qiymatini toping.

Yechish

Koshi tengsizligi (a, b, c) va $(1, 2, 2)$ uchun:

$$(a + 2b + 2c)^2 \leq (a^2 + b^2 + c^2) \cdot (1^2 + 2^2 + 2^2)$$

$$\leq 6 \cdot 9 = 54$$

Demak:

$$a + 2b + 2c \leq \sqrt{54} = 3\sqrt{6}$$

Javob

$$3\sqrt{6}$$

12-misol

Isbotlang $(a, b > 0)$:

$$a^2/b + b^2/a \geq a + b$$

Yechish

Koshi tengsizligi:

$$(a^2/b + b^2/a) \cdot (a + b) \geq (a + b)^2$$

$(a + b) > 0$ bo‘lgani uchun:

$$a^2/b + b^2/a \geq a + b$$

Xulosa

Isbotlandi.

13-misol

Agar $a^2 + b^2 = 13$ bo‘lsa,
$3a + 2b$ ning maksimal qiymatini toping.

Yechish

Koshi tengsizligi:

$$(3a + 2b)^2 \leq (a^2 + b^2) \cdot (3^2 + 2^2)$$

$$\leq 13 \cdot 13 = 169$$

Demak:

$$3a + 2b \leq 13$$

Javob

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14-misol

Isbotlang:

$$a^2 + b^2 + c^2 \geq (ab + bc + ca)$$

Yechish

Ma’lum tengsizlikdan:

$$(a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0$$

Ochamiz:

$$2(a^2 + b^2 + c^2) \geq 2(ab + bc + ca)$$

Shundan:

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

Xulosa

Isbotlandi.

15-misol

Agar $a^2 + b^2 + c^2 = 3$ bo‘lsa,
$ab + bc + ca$ ning maksimal qiymatini toping.

Yechish

Ma’lumki:

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$$

Koshi tengsizligidan:

$$(a + b + c)^2 \leq 3 \cdot 3 = 9$$

Demak:

$$9 \geq 3 + 2(ab + bc + ca)$$

$$2(ab + bc + ca) \leq 6$$
$$ab + bc + ca \leq 3$$

Javob

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