Mustaqil ishlash uchun masalalar

1-masala:

$a, b, c \ > \ 0$ bo‘lsin. Isbotlang:

$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \ 3.$

Yechim:

1) AM–GM (o’rta arifmetik-o’rta geometrik) tengsizligini qo‘llaymiz:

$\frac{x+y+z}{3} \geq \ \sqrt[3]{xyz}.$

Bu yerda:

$$x \ = \frac{a}{b}$$

$$y \ = \frac{b}{c}$$

$$z \ = \frac{c}{a}$$

2) AM–GM ni qo‘llaymiz:

$\frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}{3} \geq \ \sqrt[3]{\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a}}$

Soddalashtiramiz:

$\left(\frac{a}{b}\right) \cdot \left(\frac{b}{c}\right) \cdot \left(\frac{c}{a}\right) \ = \ 1 \ \rightarrow \sqrt[3]{1} = 1$

Demak:

$\frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}{3} \geq \ 1.$

3) Ikkala tomonni 3 ga ko‘paytiramiz:

$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \; 3.$

4) Tenglik faqat $a = b = c$ bo‘lganda bajariladi.

2-masala.

$a, b, c > 0$ bo‘lsin. Isbotlang:

$$\frac{a^2}{b + c} + \frac{b^2}{c + a} + \frac{c^2}{a + b} \geq \frac{a + b + c}{2}.$$

YECHIM

1-bosqich. Koshi tengsizligining quyidagi ko‘rinishini ishlatamiz:

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \frac{x_3^2}{y_3} \geq \frac{(x_1 + x_2 + x_3)^2}{y_1 + y_2 + y_3}$$

Bu yerda quyidagicha tanlaymiz:

$$\begin{aligned} x_1 &= a, y_1 = b + c \\ x_2 &= b, y_2 = c + a \\ x_3 &= c, y_3 = a + b \end{aligned}$$

Shunda chap tomoni aynan bizning ifodamiz bo‘ladi:

$$\frac{a^2}{b + c} + \frac{b^2}{c + a} + \frac{c^2}{a + b}$$

Demak, Koshi bo‘yicha:

$$\frac{a^2}{b + c} + \frac{b^2}{c + a} + \frac{c^2}{a + b} \geq \frac{(a + b + c)^2}{(b + c) + (c + a) + (a + b)}.$$

2-bosqich. Endi pastdagi yig‘indini soddalashtiramiz:

$$(b + c) + (c + a) + (a + b) = b + c + c + a + a + b = 2(a + b + c)$$

Shuning uchun:

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \geq \frac{(a+b+c)^2}{2(a+b+c)}$$

$a+b+c > 0$ bo‘lgani uchun, qisqartirish mumkin:

$$\frac{(a+b+c)^2}{2(a+b+c)} = \frac{a+b+c}{2}$$

Demak, natijada:

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \geq \frac{a+b+c}{2}.$$

3-bosqich. Tenglik qachon bo‘ladi?

Koshi tengsizligida tenglik bo‘lishi uchun quyidagisi kerak:

$$\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$$

Bu holat $a = b = c$ bo‘lganda bajariladi.

Demak:

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \geq \frac{a+b+c}{2}.$$

tenglik faqat $a = b = c$ bo‘lsa.

3-masala.
$a, b, c > 0$ bo‘lsin. Isbotlang:

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \geq \frac{9}{2(a+b+c)}.$$

YECHIM

1-bosqich. Koshi tengsizligining quyidagi ko‘rinishini ishlatamiz:

$$\frac{x_1^2}{y_1}+\frac{x_2^2}{y_2}+\frac{x_3^2}{y_3} \geq \frac{(x_1+x_2+x_3)^2}{y_1+y_2+y_3}.$$

Biz shunday tanlaymiz:

$$x_1 = 1, \qquad y_1 = a + b$$
$$x_2 = 1, \qquad y_2 = b + c$$
$$x_3 = 1, \qquad y_3 = c + a$$

Shunda chap tomoni bo‘ladi:

$$\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a}$$

Demak, Koshi bo‘yicha:

$$\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \geq \frac{(1 + 1 + 1)^2}{(a + b) + (b + c) + (c + a)}.$$

2-bosqich. Endi pastdagi yig‘indini soddalashtiramiz:

$$(a + b) + (b + c) + (c + a) = a + b + b + c + c + a = 2(a + b + c)$$

Shuning uchun:

$$\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \geq \frac{9}{2(a + b + c)}.$$

Bu aynan isbotlash kerak bo‘lgan tengsizlik.

Demak:

$$\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \geq \frac{9}{2(a + b + c)}.$$

3-bosqich. Tenglik qachon bajariladi?

Koshi tengsizligida tenglik bo‘lishi uchun:

$$\frac{x_1}{\sqrt{y_1}}, \frac{x_2}{\sqrt{y_2}}, \frac{x_3}{\sqrt{y_3}}$$

bir xil bo‘lishi kerak. Bizda:

$$x_1 = x_2 = x_3 = 1,$$
$$y_1 = a + b, \qquad y_2 = b + c, \qquad y_3 = c + a.$$

Demak tenglik uchun:

$$a + b = b + c = c + a$$

bo‘lishi kerak. Bu tengliklar sistemasi faqat:

$$a = b = c$$

bo‘lganda bajariladi.

Shuning uchun tenglik faqat $a = b = c$ bo‘lganda erishiladi.

MASALA (Schur + AM–GM kombinatsiyasi)

4-masala.
$a, b, c > 0$ bo‘lsin. Isbotlang:

1. $a^3 + b^3 + c^3 + 3abc \geq a^2 b + b^2 c + c^2 a + ab^2 + bc^2 + ca^2$
2. $a^2 b + b^2 c + c^2 a + ab^2 + bc^2 + ca^2 \geq 6abc$

Shundan xulosa chiqadi:

$a^3 + b^3 + c^3 + 3abc \geq 6abc.$

Bu zanjirli tengsizlik **Shur tengsizligi** va **AM–GM**ni birga ishlatish uchun yaxshi misol.

YECHIM

1-qism. Shur yordamida birinchi tengsizlik

Schur tengsizligining 3-darajali ko‘rinishi shunday:

$$a^3 + b^3 + c^3 + 3abc \geq a^2 b + a^2 c + b^2 c + b^2 a + c^2 a + c^2 b.$$

O‘ng tomonni shunday yozish mumkin:

$$a^2 b + b^2 c + c^2 a + ab^2 + bc^2 + ca^2.$$

Demak, Shur tengsizligi aynan shuni beradi:

$a^3 + b^3 + c^3 + 3abc \geq a^2 b + b^2 c + c^2 a + ab^2 + bc^2 + ca^2.$

Bu 1-qismning isboti.

Metodik eslatma: biz Shur tengsizligini allaqachon alohida isbot qilib chiqqan edik ($a = c + x, b = c + y$ o‘rin almashtirish orqali). Shu yerda esa uni **tayyor vosita** sifatida ishlatyapmiz.

2-qism. AM–GM yordamida ikkinchi tengsizlik

Endi isbotlaymiz:

$$a^2 b + b^2 c + c^2 a + ab^2 + bc^2 + ca^2 \geq 6abc.$$

Ifodani 3 juftlikka ajratamiz:

$$(a^2 b + ab^2) + (b^2 c + bc^2) + (c^2 a + ca^2).$$

Endi har bir qavs ichiga AM–GM (arifmetik va geometrik o‘rtacha) tengsizligini qo‘llaymiz.

1. Birinchi juftlik uchun:

$$a^2 b + ab^2 \geq 2\sqrt{a^2 b \cdot ab^2} = 2\sqrt{a^3 b^3} = 2ab\sqrt{ab}.$$

Lekin bizga aniqroq va sodda baho kerak. Yana bir yo‘l:

$a^2 b + ab^2 = ab(a + b).$

Bu ifoda ijobiy va simmetrik, lekin biz aniq $\geq 2abc$ bahosini istaymiz. Buni quyidagicha qilamiz:

AM–GM ni **to‘g‘ridan-to‘g‘ri** abc ga bog‘laymiz:

Har bir juftlik uchun shunday yozamiz:

$a^2 b + ab^2 \geq 2ab\sqrt{ab}$ (bu to‘g‘ri, lekin abc bilan bog‘lash uchun maxsus shart kerak bo‘ladi)

Bu yerdan oddiyroq va eng ko‘p ishlatiladigan usul:

Har bir juftlikni quyidagicha baholaymiz:

$$a^2 b + ab^2 \geq 2ab\sqrt{ab} \geq 2abc$$

agar, masalan, $c \leq \sqrt{ab}$ bo‘lsa. Lekin bu har doim shart emas. Shu sababli **oddiyroq va standart yondashuv**ni tanlaymiz:

Biz quyidagi klassik AM–GM juftlashini ishlatamiz:

$$a^2\,b\ +\ bc^2 \geq\ 2\,a\,b\,c$$
$$b^2\,c\ +\ ca^2 \geq\ 2\ a\,b\,c$$
$$c^2\,a\ +\ ab^2 \geq\ 2\,a\,b\,c.$$

Bu yerda har qaysi juftlikda uchta harf qatnashadi. Masalan:

- $a^2\,b\ +\ bc^2$ uchun:
$$x\ =\ a^2\,b, y\ =\ b\,c^2$$
AM–GM bo‘yicha:
$a^2\,b\ +\ b\,c^2 \geq\ 2\sqrt{a^2\,b\ \cdot\ b\,c^2}\ =\ 2\sqrt{a^2\,b^2\,c^2}\ =\ 2abc.$

Shu kabi:

- $b^2\,c\ +\ ca^2 \geq\ 2abc$
- $c^2\,a\ +\ ab^2 \geq\ 2abc.$

Endi ularni qo‘shamiz:

$$(a^2\,b\ +\ bc^2)\ +\ (b^2\,c\ +\ ca^2)\ +\ (c^2\,a\ +\ ab^2) =\ 2abc\ +\ 2abc\ +\ 2abc =\ 6abc.$$

Lekin chap tomonni tartiblashimiz kerak:

$$a^2\,b\ +\ b^2\,c\ +\ c^2\,a\ +\ ab^2\ +\ bc^2 +\ ca^2$$

bu aynan yuqoridagi uch juftlikning yig‘indisiga teng.

Demak:

$$a^2b\ +\ b^2c\ +\ c^2a\ +\ ab^2\ +\ bc^2\ +\ ca^2 \geq\ 6abc.$$

Bu 2-qismning isboti.

Yakuniy zanjir

Endi ikkala natijani birlashtiramiz:

1. Shur bo‘yicha:
$$a^3\ +\ b^3\ +\ c^3\ +\ 3abc \geq\ a^2\,b\ +\ b^2\,c\ +\ c^2\,a\ +\ ab^2\ +\ bc^2\ +\ ca^2.$$
2. AM–GM bo‘yicha:
$$a^2\,b\ +\ b^2\,c\ +\ c^2\,a\ +\ ab^2\ +\ bc^2\ +\ ca^2\ \geq\ 6abc.$$

Shunday qilib:

$$a^3 + b^3 + c^3 + 3abc \geq 6abc.$$

5-masala.
$a, b, c > 0$ bo‘lsin. Isbotlang:

$$\left(\frac{a}{b + c}\right)^2 + \left(\frac{b}{c + a}\right)^2 + \left(\frac{c}{a + b}\right)^2 \geq 3/4.$$

Ya’ni:

$$\frac{a^2}{(b + c)^2} + \frac{b^2}{(c + a)^2} + \frac{c^2}{(a + b)^2} \geq 3/4.$$

YECHIM

1-bosqich. Koshi (yoki RMS–AM) usulini ishlatamiz.

Har qanday uchta haqiqiy son x, y, z uchun:

$\frac{x^2 + y^2 + z^2}{3} \geq \left(\frac{x + y + z}{3}\right)^2.$

Bu tengsizlikdan:

$x^2 + y^2 + z^2 \geq \frac{(x + y + z)^2}{3}$.

Endi bu yerda:

$$x = \frac{a}{b + c}$$
$$y = \frac{b}{c + a}$$
$$z = \frac{c}{a + b}$$

deb olamiz.

Shunda:

$$\left(\frac{a}{b + c}\right)^2 + \left(\frac{b}{c + a}\right)^2 + \left(\frac{c}{a + b}\right)^2 = \frac{\left(\frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b}\right)^2}{3}.$$

Demak:

$$\frac{a^2}{(b+c)^2} + \frac{b^2}{(c+a)^2} + \frac{c^2}{(a+b)^2} = \frac{(S)^2}{3},$$

bu yerda:

$S = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}.$

2-bosqich. Endi S uchun pastki chegarani ishlatamiz (Nesbitt tengsizligi).

Nesbitt tengsizligi shuni aytadi:

$$a, b, c > 0 \rightarrow \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Demak:

$S \geq \frac{3}{2}.$

3-bosqich. S ni qayta qo‘yamiz.

1-bosqichdagi natijaga $S \geq \frac{3}{2}$ ni qo‘yamiz:

$$\frac{a^2}{(b+c)^2} + \frac{b^2}{(c+a)^2} + \frac{c^2}{(a+b)^2} = \frac{S^2}{3} = \frac{\left(\frac{3}{2}\right)^2}{3} = \frac{\frac{9}{4}}{3} = \frac{9}{12} = \frac{3}{4}.$$

Demak:

$\left(\frac{a}{b+c}\right)^2 + \left(\frac{b}{c+a}\right)^2 + \left(\frac{c}{a+b}\right)^2 \geq \frac{3}{4}.$

4-bosqich. Tenglik qachon bo‘ladi?

Ikki joyda tenglik ishlatilgan:

1. $x^2 + y^2 + z^2 \geq \frac{(x+y+z)^2}{3}$
   tenglik faqat $x = y = z$ bo‘lganda.
2. Nesbitt tengsizligida
   $$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$
   tenglik faqat $a = b = c$ bo‘lganda.

Demak, barcha tengliklar bir vaqtda faqat:

$$a = b = c$$

bo‘lganda bajariladi.