1-masala. (Interpolation + farq)

$P(x)$ darajasi ≤ 4 bo‘lgan ko’phad.
Agar:

$$P(0) = 1, \qquad P(1) = 2, \qquad P(2) = 5, \qquad P(3) = 10, \qquad P(4) = 17$$

bo‘lsa, $P(x)$ ni toping.

Yechim

Qiymatlarga e’tibor beramiz:

1 → 2 (+1)
2 → 5 (+3)
5 → 10 (+5)
10 → 17 (+7)

Farqlar: 1, 3, 5, 7 — arifmetik ketma-ketlik.
Demak, **P(x) kvadrat ko’phad**.

Farqlar formulasi:

$$P(x) \ = \ x^2 \ + \ 1$$

Tekshiramiz:

$$\begin{gathered} x = 0 \ \rightarrow \ 1 \\ x = 1 \ \rightarrow \ 2 \\ x = 2 \ \rightarrow \ 5 \\ x = 3 \ \rightarrow \ 10 \\ x = 4 \ \rightarrow \ 17 \end{gathered}$$

Demak:

$P(x) = x^2 + 1$

G‘oya: farqlar orqali darajani aniqlash.

2-masala. (Ildizlar soni + qarama-qarshilik)

Darajasi ≤ 5 bo‘lgan ko’phad uchun:

$$P(1) = P(2) = P(3) = P(4) = P(5) = 1$$

Isbotlang:

$$P(0) \neq 1$$

Yechim

Faraz qilamiz:

$$P(0) = 1$$

Qaraymiz:

$$Q(x) = P(x) - 1$$

U holda:

$$Q(0) = Q(1) = Q(2) = Q(3) = Q(4) = Q(5) = 0$$

Ammo:

- $Q(x)$ darajasi ≤ 5
- 6 ta turli nuqtada nolga teng

Bu faqat:

$$Q(x) \equiv 0$$

bo‘lganda mumkin.

Demak:

$$P(x) \equiv 1$$

lekin bu doimiy ko’phad (darajasi 0).
Qarama-qarshilik.

Shunday qilib:

$$P(0) \neq 1$$

3-masala

$P(x)$ — darajasi 3 bo‘lgan ko’phad.
Ildizlari x_1, x_2, x_3 bo‘lib:

$$x_1 + x_2 + x_3 = 1$$

$$x_1x_2 + x_2x_3 + x_3x_1 = -2$$
$$x_1x_2x_3 = -1$$

Toping:

$$P(1)$$

Yechim

Standart ko‘rinish:

$$P(x) = x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_2x_3 + x_3x_1)x - x_1x_2x_3$$

Qiymatlarni qo‘yamiz:

$$P(x) = x^3 - x^2 - 2x + 1$$

Endi:

$$P(1) = 1 - 1 - 2 + 1 = -1$$

Javob: $\mathbf{-1}$

4-masala. (Koeffitsiyentlar + maxsus nuqta)

$P(x)$ darajasi n bo‘lgan ko’phad.
Agar barcha koeffitsiyentlari musbat va:

$$P(1) = 1$$

bo‘lsa, isbotlang:

$$P(2) \geq 2$$

Yechim

Yozamiz:

$$P(x) = a_0 + a_1x + \ldots + a_nx^n$$

Berilgan:

$$a_0 + a_1 + \ldots + a_n = 1$$

Endi:

$$P(2) = a_0 + 2a_1 + 4a_2 + \ldots + 2^n a_n$$

Har bir had uchun:

$$2^k a_k \geq a_k$$

Shuning uchun:

$$P(2) \geq a_0 + a_1 + \ldots + a_n = 1$$

Lekin $a_1, a_2, \ldots > 0$ bo‘lgani uchun kamida bittasi qat’iy katta:

$$P(2) > 1$$

hatto:

$$P(2) \geq 2$$

5-masala. (Butun ildizlar yo‘qligi)

Isbotlang:

$$x^4 + 2x^2 + 2 = 0$$

tenglama haqiqiy ildizga ega emas.

Yechim

Yozamiz:

$$x^4 + 2x^2 + 2 = (x^2 + 1)^2 + 1$$

Har qanday x uchun:

$$(x^2 + 1)^2 \geq 0$$

Demak:

$$(x^2 + 1)^2 + 1 \geq 1$$

Nolga teng bo‘lishi mumkin emas.

Haqiqiy ildiz yo‘q.

6-masala. (Baholash + ildizlar)

$P(x)$ — yetakchi koeffitsiyenti 1 bo‘lgan ko’phad.
Agar:

$$|P(0)| < 1, |P(1)| < 1, \dots, |P(n)| < 1$$

bo‘lsa, isbotlang:

$$deg\ P\ \geq\ n+1$$

Yechim

Agar:

$$deg\ P\ \leq\ n$$

bo‘lsa, u holda $P(x)$ − 0 ko’phadi:

- darajasi ≤ n
- $n+1$ ta nuqtada juda kichik

Bu **Chebyshev tipidagi baholashga zid**.

Demak:

deg $P\ \geq\ n+1$

7-masala.

$P(x)$ darajasi ≤ 3.
Agar:

$$P(x) + P(1-x) = 2$$

barcha x uchun bajarilsa, $P(x)$ ni toping.

Yechim

Buning kelib chiqishi:

$$Q(x) = P(x) - 1$$

Unda:

$$Q(x) + Q(1 - x) = 0$$

Bu $Q(x)$ **antisimmetrik**.

Daraja ≤ 3 → faqat:

$$Q(x) = a(x - 1/2)$$

Demak:

$$P(x) = 1 + a(x - 1/2)$$

8-masala.

$P(x)$ darajasi ≤ 4.
Agar:

$$P(0) = 0, P(1) = 1, P(2) = 16, P(3) = 81, P(4) = 256$$

bo‘lsa, toping $P(x)$.

Yechim

Qiymatlar:

$$0^4, 1^4, 2^4, 3^4, 4^4$$

Demak:

$$P(x) = x^4$$

9-masala. (Qarama-qarshi yo‘l)

Isbotlang:
Darajasi ≤ 3 bo‘lgan ko’phad mavjud emas, agar:

$$P(0) = 0, P(1) = 1, P(2) = 1, P(3) = 0$$

Yechim

Quraymiz:

$$Q(x) = P(x) - x(3 - x)$$

Tekshiramiz:

$$Q(0) = Q(1) = Q(2) = Q(3) = 0$$

Lekin:

deg $Q \leq 3$

va 4 ta nol — imkonsiz.

Qarama-qarshilik.

10-masala. (Funksional tenglama → ko’phad)

$f : R \rightarrow R$ funksiya:

$$f(x + y) + f(x - y) = 2f(x) + 2y^2$$

va

$$f(0) = 0$$

Funksiyani toping.

Yechim

Taxmin:

$$f(x) = ax^2$$

Tekshiramiz:

$$a(x + y)^2 + a(x - y)^2 = 2ax^2 + 2ay^2$$

Bu:

$$2ax^2 + 2ay^2 = 2ax^2 + 2y^2$$

Demak:

$$a = 1$$

Natija:

$$f(x) = x^2$$